3.2 INTERACTION AND I/O PAIRING

3.2.1 INTERACTION

- When $G_{c1}$ is open, the process that $G_{c2}$ controls is $G_{22}$.

- When $G_{c1}$ is closed, however, the process controlled by $G_{c2}$ becomes

$$\tilde{G}_{22} = G_{22} - \frac{G_{21}G_{c1}G_{12}}{1 + G_{c1}G_{11}}$$

If both $G_{12}$ and $G_{21}$ are not zero, $\tilde{G}_{22}$ varies with $G_{c1}$.

If $G_{c1}$ is adjusted, $G_{c2}$ should be retuned, too.

- Same thing can be said for $G_{c1}$.

- The above problem is caused by the interaction through $G_{21}$ and $G_{12}$.
Ex. Consider the following process:

\[
\begin{bmatrix}
  y_1 \\
  y_2 \\
\end{bmatrix}
= 
\begin{bmatrix}
  G_{11} & G_{12} \\
  G_{21} & G_{22} \\
\end{bmatrix}
\begin{bmatrix}
  u_1 \\
  u_2 \\
\end{bmatrix}
= 
\begin{bmatrix}
  \frac{2}{10s+1} & \frac{1.5}{s+1} \\
  \frac{1.5}{s+1} & \frac{2}{10s+1} \\
\end{bmatrix}
\begin{bmatrix}
  u_1 \\
  u_2 \\
\end{bmatrix}
\]

Assume that P-control is used for both \( G_{c1} \) and \( G_{c2} \).

- When only one of \( y_1 \) and \( y_2 \) is under control, the controller gain can have value (as far as it is positive) without causing stability problem.
- When both control loops are closed, stability is attained for the controller gains in a limited region shown below. Both gains cannot be increased simultaneously.
Ex. This time, we consider the process

\[
\begin{bmatrix}
    y_1 \\
    y_2 
\end{bmatrix} = \begin{bmatrix}
    G_{11} & G_{12} \\
    G_{21} & G_{22}
\end{bmatrix} \begin{bmatrix}
    u_1 \\
    u_2
\end{bmatrix} = \begin{bmatrix}
    \frac{2}{10s+1} & \frac{1.5}{s+1} \\
    -\frac{1.5}{s+1} & \frac{2}{10s+1}
\end{bmatrix} \begin{bmatrix}
    u_1 \\
    u_2
\end{bmatrix}
\]

Again, it is assumed that P-control is used for both \( G_{c1} \) and \( G_{c2} \).

- The stability region is changed to

![Stability Region Diagram](image-url)

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3.2.2 I/O PAIRING

- Suppose that we want to control a MIMO process using single-loop controllers (in a decentralized fashion).

- Then the pairing will be the primary question. $u_1 - y_1 / u_2 - y_2$ or $u_1 - y_2 / u_2 - y_1$?

- From the previous considerations, the judgement for proper pairing can be made based on an interaction measure.

- How to measure interaction? Many different ways, but the most widely used one is the relative gain by Bristol(1968).

Relative Gain

![Relative Gain Diagram]

\[ \Delta u_1 \quad \text{Delta} \]

\[ \begin{align*}
  G_1 & \quad \rightarrow \quad G_{11} \\
  G_{21} & \quad \rightarrow \quad G_{12} \\
  G_{22} & \quad \rightarrow \quad y_1 \\
  y_1 & \quad \rightarrow \quad \Delta y_1^{op} \\
  \Delta y_1^{cl} & \quad \rightarrow \quad \text{Delta} \\
  G_2 & \quad \rightarrow \quad G_{21} \\
  G_{12} & \quad \rightarrow \quad G_{22} \\
  y_2 & \quad \rightarrow \quad \text{Delta} \\
  u_2 & \quad \rightarrow \quad \text{Delta} \\
  r_2 & \quad \rightarrow \quad \text{Delta} \\
\end{align*} \]
Definition:

\[ \lambda_{11} = \frac{\Delta y_1^{op}/\Delta u_1}{\Delta y_1^{cl}/\Delta u_1} \]

The relative gain is usually defined under steady state conditions.

- The relative gain is usually defined under steady state conditions.
- \( \Lambda = [\lambda_{ij}] \) is called the Relative Gain Array.

Interpretations:

- Obviously, \( \lambda_{11} = 1 \) when \( G_{12} \) and/or \( G_{21} \) is zero. \( \Rightarrow \) No interaction, \( u_1 - y_1 \) pair is decoupled from other loops.
- \( \lambda_{11} = 0 \) when \( G_{11} = 0 \) \( \Rightarrow \) No coupling between \( u_1 \) and \( y_1 \); \( y_1 \) should be paired with \( u_2 \) for a \( 2 \times 2 \) process.
- \( \lambda_{11} > 1 \) \( \Rightarrow \) The gain is increased when other loops are closed.
- \( \lambda_{11} < 1 \) \( \Rightarrow \) Sign of the gain is reversed when other loops are closed.
- \( \lambda_{11} \gg 1 \) or \( \lambda_{11} \ll 1 \) implies that the system has serious interaction. \( \rightarrow \) SISO pairing has limitations. \( \rightarrow \) should rely on MIMO control.
Properties:

- For an $n \times n$ process, $\sum_i \lambda_{ij} = \sum_j \lambda_{ij} = 1$.
  For a $2 \times 2$ process,

$\lambda_{11} = \frac{1}{1 - K_{12}K_{21}/K_{11}K_{22}}$, \quad \lambda_{22} = \lambda_{11}, \quad \lambda_{12} = \lambda_{21} = 1 - \lambda_{11}$

- The relative gain can be directly computed from the steady state gain matrix of the process.

Let $K$ be the steady state gain matrix of a process. Then,

$\Lambda = K \otimes K^{-T}$

where $\otimes$ denotes the element-by-element multiplication.
3.3 **DECOUPLING**

- When $\lambda_{11}$ is far from one for a $2 \times 2$ process, decentralized control has limitations. $\Rightarrow$ Multivariable Control
- One of the classical multivariable control techniques is *decoupling control*.

**Decoupled Process**

If we neglect the input saturation blocks,

\[
\begin{align*}
y_1 &= (G_{11} + G_{12}D_{12})m_1 + (G_{12} + G_{11}D_{21})m_2 \\
y_2 &= (G_{21} + G_{22}D_{12})m_1 + (G_{22} + G_{21}D_{21})m_2
\end{align*}
\]
Let

\begin{align*}
D_{21} &= -\frac{G_{12}}{G_{11}} \\
D_{12} &= -\frac{G_{21}}{G_{22}}
\end{align*}

Then

\begin{align*}
y_1 &= \left( G_{11} - \frac{G_{12}G_{21}}{G_{22}} \right) m_1 = G'_{11} m_1 \\
y_2 &= \left( G_{22} - \frac{G_{21}G_{12}}{G_{11}} \right) m_2 = G'_{22} m_2
\end{align*}

The relative gain for the decoupled process, \( \lambda_{11d} \) is one.
- Multivariable Control

\[
\begin{align*}
u_1 &= m_1 + D_{21}m_2 \\
u_2 &= m_2 + D_{12}m_1
\end{align*}
\]

Remarks:

- A decisive drawback of decoupling control is that it is sensitive to model error.
  - For processes with \( \lambda_{11} \approx 1, \lambda_{11d} \approx 1 \).
  - As \( \lambda_{11} \) deviates from one, \( \lambda_{11d} \) also deviates from one.

Let

\[
D_{21} = (1 + \delta) \left( -\frac{G_{12}}{G_{11}} \right)
\]

\[
D_{12} = (1 + \delta) \left( -\frac{G_{21}}{G_{22}} \right)
\]
• The decoupler can be designed at the output.

• To design decoupling control, *Process Model* is required! Why not try other MIMO control techniques?