3.4 CONVEX OPTIMIZATION

Convexity

Convex set: \( C \subseteq \mathbb{R}^n \) is convex if

\[
x, y \in C, \ \lambda \in [0, 1] \Rightarrow \lambda x + (1 - \lambda)y \in C
\]

Convex Functions: \( f : \mathbb{R}^n \to \mathbb{R} \) is convex if

\[
x, y \in \mathbb{R}^n, \ \lambda \in [0, 1] \\
f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)
\]
Convexity (Continued)

Notice that \( \{ x : g(x) \leq 0 \} \) is convex if \( g \) is convex.

Theorem: If \( f \) and \( g \) are convex any local optimum is globally optimal.
Linear Programs

$$\min_{x \in \mathbb{R}^n} a^T x$$

subject to

$$Bx \leq b$$

Linear program is a convex program.

Feasible basic solution: feasible solution that satisfies \( n \) of the constraints as equalities.

Fact: If an optimal solution exists, there exists a feasible basic solution that is optimal.
Quadratic Programs

\[ \min_{x \in \mathbb{R}^n} \frac{1}{2} x^T H x + g^T x \]

subject to

\[ Ax \leq b \]

Quadratic program is convex if \( H \) is positive semi-definite.
3.5 ALGORITHMS FOR CONSTRAINED OPTIMIZATION PROBLEMS

Algorithms for Linear Program

Simplex Method

Motivation: There always exists a basic optimal solution.

Main Idea:

- Find a basic solution.
- Find another basic solution with lower cost function value.
- Continue until another basic solution with lower cost function value cannot be found.

Simplex algorithm always finds a basic optimal solution.
Algorithms for Linear Program (Continued)

Interior Point Method

Main Idea:

- Define barrier function:

\[ B = - \sum_{i=1}^{m} \frac{1}{c_i^T x - b_i} \]

- Form the unconstrained problem:

\[ \min_{x} a^T x + \frac{1}{K} B(x) \]

- Solve the unconstrained problem using Newton method.

- Increase \( K \) and solve the unconstrained problem again until the solution converges.

- Remarkably, problems seem to converge between 5 to 50 Newton steps regardless of the problem size.

- Can exploit structures of the problem (e.g. sparsity) to reduce computation time per Newton step.

- Can be extended to general nonlinear convex problems such as quadratic programs.
Algorithms for Quadratic Program

Active Set Method

Main Idea:

- Determine the active constraints and set them as equality constraints.
- Solve the resulting problem.
- Check the Kuhn-Tucker condition that is also sufficient for QP.
- If Kuhn-Tucker condition is not satisfied, try another set of active constraints.

Interior Point Method

- The main idea of interior point method for QP is the same as that for LP.
Generalized Reduced Gradient Method
for Constrained Nonlinear Programs

Main idea:

1. Linearize the equality constraints that are possibly obtained adding slack variables
2. Solve the resulting linear equations for \( m \) variables
3. Apply the steepest descent method with respect to \( n - m \) variables

Linearization of Constraints:

\[
\nabla_y h(y, z) dy + \lambda^T \nabla_z h(y, z) dz = 0
\]

\[
\downarrow
\]

\[
dy = -[\nabla_y h(y, z)]^{-1} \lambda^T \nabla_z h(y, z) dz
\]

Generalized Reduced Gradient of Objective Function:

\[
df(y, z) = \nabla_y f(y, z) dy + \lambda^T \nabla_z f(y, z) dz
\]

\[
= [\lambda^T \nabla_z f(y, z) - \nabla_y f(y, z)[\nabla_y h(y, z)]^{-1} \lambda^T \nabla_z h(y, z)] dz
\]

\[
\downarrow
\]

\[
r = \frac{df}{dz} = \lambda^T \nabla_z f(y, z) - \nabla_y f(y, z)[\nabla_y h(y, z)]^{-1} \lambda^T \nabla_z h(y, z)
\]
Penalty Method for Constrained Nonlinear Programs

Main idea: Instead of forcing the constraints, penalize the violation of the constraints in the objective.

$$\min_x f(x) - c_k g(x) \quad (P_k)$$

where $c_k > 0$.

Theorem: Let $x_k$ be the optimal solution of $(P_k)$. Then as $c_k \to \infty$, $x_k \to x^*$. 
Successive QP Method
for Constrained Nonlinear Programs

Main idea:

1. Approximate the object function by quadratic function and constraints linear function.

2. Solve the resulting quadratic problem

Approximate Quadratic Program:

\[
\min \nabla f d x + \frac{1}{2} d x ^ { T } \nabla ^ { 2 } f d x \\
\text{subject to} \quad g(x) + \nabla g(x) dx \leq 0
\]
Nonconvex Programs

The aforementioned optimization algorithms indentify only one local optimum.

However, a nonconvex optimization problem may have a number of local optima.

\[ \downarrow \]

Algorithms that identifies a global optimum are necessary
A Global Optimization Algorithm
for Noconvex Programs

Branch and bound type global optimization algorithm:

- Branching Step: split the box at the optimum
- Bounding Step: find the box where the optimum is lowest